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On Turán Quadrature Formulas for the Chebyshev Weight*

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As we know, the Chebyshev weight $w(x)=(1-x^2)^{-1/2}$ has the property: For each fixed n, the solutions of the extremal problem $\int_{-1}^{1}[\prod_{k=1}^{n}(x-x_k)]^m\,w(x)\,dx=\min_{P=x^n+\cdots}\int_{-1}^{1}P(x)^m\,w(x)\,dx$ for every even m are the same. This paper proves that the Chebyshev weight is the only weight having this property (up to a linear transformation).

1. INTRODUCTION AND MAIN RESULTS

Let w be a weight (function) on $\mathbb{R}$ satisfying that $w(x)=0$ for $|x|>1$ and $\int_{-1}^{1}w(x)\,dx=1$ and let $\Delta(w)$ denote the smallest closed interval such that $\int_{\Delta(w)}w(x)\,dx=1$. Denote by $\mathbf{P}_N$ the set of polynomials of degree at most N. According to Theorem 4 in [2], for even $m\in\mathbb{N}$ if $\omega_n(x):=\prod_{k=1}^{n}(x-x_k)$ with

$$-1\leqslant x_1<x_2<\cdots<x_n\leqslant 1 \tag{1.1}$$

satisfies

$$\int_{-1}^{1}\omega_n(x)^m\,w(x)\,dx=\min_{P=x^n+\cdots}\int_{-1}^{1}P(x)^m\,w(x)\,dx, \tag{1.2}$$

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then the quadrature formula with certain numbers $c_{ikm} := c_{ikmn}$ (called Cotes numbers of higher order)

$$\int_{-1}^{1} f(x)\, w(x)\, dx = \sum_{i=0}^{m-2} \sum_{k=1}^{n} c_{ikm} f^{(i)}(x_k) \tag{1.3}$$

is exact for all $f \in \mathbf{P}_{mn-1}$.

As Turán pointed out in [8, p. 46], particularly interesting is the Chebyshev weight

$$w(x) = \frac{1}{\pi \sqrt{1-x^2}}. \tag{1.4}$$

By a theorem of Bernstein [1], in this case the nth Chebyshev polynomial of first kind $2^{1-n}T_n(x)$ is the solution of (1.2) for all even $m \in \mathbb{N}$. This elegant property is very useful, say, this makes it possible to give an explicit formula for the Cotes numbers c_{ikm} [7]. Examples of other weights for which the solutions of (1.2) are independent of m (but vary with n) can be found in the recent paper [4] given by Gori and Miccelli. It is natural to ask whether or not there are other weights having this property. Clearly, a linear transformation of the weight (1.4)

$$w(x) = v_{\alpha,\beta}(x) := \begin{cases} \dfrac{1}{\pi \sqrt{(x-\alpha)(\beta - x)}}, & x \in (\alpha, \beta), \\ 0, & x \notin (\xi, \beta), \end{cases} \tag{1.5}$$

also admits this property. The aim of this note is to prove that the Chebyshev weight (1.4) is the only weight (up to a linear transformation). In fact, we shall prove slightly more:

THEOREM. *Let w be a weight supported in $[-1, 1]$ such that $\int_{-1}^{1} w(x)\, dx = 1$. If the formula (1.2) holds for the following pairs (m, n):*

$$m = \begin{cases} m_1, m_2, ..., & \textit{if} \quad n = 1, 2, 4, \\ 2, 4, & \textit{if} \quad n = 3, 5, 6, ..., \end{cases} \tag{1.6}$$

where $\{m_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of even natural numbers such that $m_1 = 2$ and

$$\sum_{k=1}^{\infty} \frac{1}{m_k} = \infty, \tag{1.7}$$

then there exists two numbers α and β such that $w = v_{\alpha,\beta}$.

This paper is organized as follows: In Section 2 some auxiliary lemmas are provided and in Section 3 the proof of the theorem is given.

2 AUXILIARY LEMMAS

LEMMA 1. *Let $\Delta(w)=[a,b]$ and let $\{m_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of odd natural numbers satisfying* (1.7). *If for a point $c\in(a,b)$ the relation*

$$\int_a^b (x-c)^m\, w(x)\, dx=0 \tag{2.1}$$

holds for every $m=m_k$, then (2.1) *holds for every odd $m\in\mathbb{N}$. Moreover,*

$$c=\frac{a+b}{2} \tag{2.2}$$

and

$$w(c-x)=w(c+x), \qquad a.e.$$

Proof. Clearly, (2.3) implies that (2.2) is valid in (2.1) holds for every odd $m\in\mathbb{N}$. So it is enough to prove (2.3). Let ξ and δ satisfy $0<\xi<\xi+\delta<h$, where $h=\max\{c-a, b-c\}$. Put

$$f_\delta(x)=\begin{cases}0, & x\in[0,\xi],\\ \dfrac{x-\xi}{\delta}, & x\in[\xi,\xi+\delta],\\ 1, & x\in[\xi+\delta,h]\end{cases}$$

and $f_\delta(x)=-f_\delta(-x)$, $x\in[-h,0]$. Clearly, $f_\delta\in C[-h,h]$ and f_δ is an odd function. By the Müntz Theorem [3, p. 197] it follows from (1.7) that given an arbitrary number $\varepsilon>0$ there is a polynomial of the form $P_\varepsilon(x)=\sum a_k x^{m_k}$ such that

$$|f_\delta(x)-P_\varepsilon(x)|\leqslant\varepsilon, \qquad x\in[-h,h].$$

Hence

$$|f_\delta(x-c)-P_\varepsilon(x-c)|\leqslant\varepsilon, \qquad x\in[c-h,c+h]. \tag{2.4}$$

Since (2.1) holds for every $m=m_k$, we obtain

$$\int_{c-h}^{c+h} P_\varepsilon(x-c)\, w(x)\, dx=0,$$

which, coupled with (2.4), yields

$$\left| \int_{c-h}^{c+h} f_\delta(x-c)\, w(x)\, dx \right| \leqslant \varepsilon.$$

Noting that $f_\delta(x-c)$ is independent of ε and ε is arbitrary, we have

$$\int_{c-h}^{c+h} f_\delta(x-c)\, w(x)\, dx = 0.$$

Furthermore, as $\delta \to \infty$ we get

$$\int_{c-h}^{c-\xi} w(x)\, dx = \int_{c+\xi}^{c+h} w(x)\, dx.$$

Differentiating this relation with respect to ξ gives

$$w(c-\xi) = w(c+\xi), \qquad \text{a.e.}$$

This is equivalent to (2.3). ∎

LEMMA 2. *Let $\Delta(w) = [a, b]$ and let $\{m_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of even natural numbers satisfying* (1.7). *If for a point $c \in (a, b)$ the formula*

$$\int_a^b (x-c)^m\, w(x)\, dx = \min_t \int_a^b (x-t)^m\, w(x)\, dx \tag{2.5}$$

holds for every $m = m_k$, then (2.5) *holds for every even $m \in \mathbb{N}$. Moreover,* (2.2) *and* (2.3) *are valid.*

Proof. Since (2.5) means

$$\int_a^b (x-c)^{m-1}\, w(x)\, dx = 0,$$

our conclusions follow directly from Lemma 1. ∎

LEMMA 3. *Let $\Delta(w) = [a, b]$ and let $\{m_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of even natural numbers satisfying* (1.7). *Further, assume that* (2.2) *and* (2.3) *are valid. If for a number d the formula*

$$\int_a^b [(x-c)^2 - d]^m\, w(x)\, dx = \min_t \int_a^b [(x-c)^2 - t]^m\, w(x)\, dx \tag{2.6}$$

holds for every $m = m_k$, *then* (2.6) *holds for every even* $m \in \mathbb{N}$. *Moreover,*

$$d = \frac{(b-a)^2}{8} \tag{2.7}$$

and

$$\sqrt{(x-a)(b-x)}\, w(x) = |x-c|\, w(\sqrt{(x-a)(b-x)}). \tag{2.8}$$

Furthermore, for every even function f

$$\int_a^b f(x-c)\, w(x)\, dx = \int_a^b f(\sqrt{(x-a)(b-x)})\, w(x)\, dx. \tag{2.9}$$

Proof. For simplicity we treat the special case $-a = b = 1$ (hence $c = 0$) only, because by the transformation $x = ((b-a)/2)\, y + ((b+a)/2)$ the general case will lead to this case. In this case (2.3), (2.7), (2.8), and (2.9) become

$$w(-x) = w(x), \qquad \text{a.e.}, \tag{2.10}$$

$$d = \tfrac{1}{2}, \tag{2.11}$$

$$\sqrt{1-x^2}\, w(x) = |x|\, w(\sqrt{1-x^2}), \tag{2.12}$$

and

$$\int_{-1}^{1} f(x)\, w(x)\, dx = \int_{-1}^{1} f(\sqrt{1-x^2})\, w(x)\, ds. \tag{2.13}$$

Meanwhile, under the assumption (2.10) in this case (2.6) holds if and only if

$$\int_{-1}^{1} (x^2-d)^{m-1}\, w(x)\, dx = 0. \tag{2.14}$$

It follows from (2.10) and (2.14) that

$$\int_0^1 (x^2-d)^{m-1}\, w(x)\, dx = 0.$$

By using the substitution $x = \sqrt{y}$ according to the assumptions of the lemma the relation

$$\int_0^1 (y-d)^{m-1} \frac{w(\sqrt{y})}{\sqrt{y}}\, dy = 0 \tag{2.15}$$

holds for every $m = m_k$. Applying Lemma 1 we conclude that (2.15), (2.14), and (2.6) hold for every even m. Moreover, we obtain (2.11) and

$$\frac{w(\sqrt{\frac{1}{2} - y})}{\sqrt{\frac{1}{2} - y}} = \frac{w(\sqrt{\frac{1}{2} + y})}{\sqrt{\frac{1}{2} + y}},$$

which, using the substitution $y = x^2 - \frac{1}{2}$, gives (2.12). In order to obtain (2.13) we apply (2.12) and use the substitution $x = \sqrt{1 - y^2}$:

$$\begin{aligned}
\int_{-1}^{1} f(x)\, w(x)\, dx &= 2 \int_{0}^{1} f(x)\, w(x)\, dx \\
&= 2 \int_{0}^{1} f(x) \frac{x w(\sqrt{1 - x^2})}{\sqrt{1 - x^2}}\, dx \\
&= 2 \int_{0}^{1} f(\sqrt{1 - y^2})\, w(y)\, dy \\
&= \int_{-1}^{1} f(\sqrt{1 - x^2})\, w(x)\, dx. \quad \blacksquare
\end{aligned}$$

LEMMA 4. *Let*

$$v_j = \frac{1}{\pi} \int_{-1}^{1} x^j \frac{dx}{\sqrt{1 - x^2}}, \qquad j = 0, 1, \ldots.$$

Then

$$v_{2j+1} = 0, \qquad v_{2j} = \frac{(2j)!}{2^{2j} (j!)^2}, \qquad j = 0, 1, 2, \ldots.$$

Proof. The first formula is trivial. To prove the second use the well-known formula [5, Formula 1.320–5, p. 25]

$$\cos^{2j} t = 2^{-2j} \left[\sum_{k=0}^{j-1} 2 \binom{2j}{k} \cos(2j - 2k)\, t + \binom{2j}{j} \right]$$

and obtain

$$x^{2j} = 2^{-2j} \left[\sum_{k=0}^{j-1} 2 \binom{2j}{k} T_{2j-2k}(x) + \binom{2j}{j} \right].$$

Hence

$$v_{2j} = 2^{-2j}\binom{2j}{j} = \frac{(2j)!}{2^{2j}(j!)^2}. \quad \blacksquare$$

3. PROOF OF THE THEOREM

As in the proof of Lemma 3 it is enough to treat the special case $-a = b = 1$ (hence $c = 0$) and to prove (1.4). Let

$$\mu_j = \int_{-1}^{1} x^j w(x)\, dx, \qquad j = 0, 1, ...$$

Then according to the Favard Theorem [6, Vol. 2, Chap. 8, Sec. 6] it is sufficient to establish

$$\mu_j = v_j, \qquad j = 0, 1, ... \tag{3.1}$$

We separate the cases when $j \leqslant 8$ and all other values of j.

By Lemma 2 we have (2.10) and hence

$$\mu_{2j+1} = 0, \qquad j = 0, 1, ... \tag{3.2}$$

Meanwhile by means of Lemma 3 (2.11)–(2.14) holds. It follows from (2.11) and (2.14) with $m = 2, 4, 6$ that

$$\mu_2 = d = \tfrac{1}{2}, \tag{3.3}$$

$$\mu_6 = 3d\mu_3 - 3d^2\mu_2 + d^3 = \tfrac{3}{2}\mu_4 - \tfrac{1}{4}, \tag{3.4}$$

$$\mu_{10} = 5d\mu_8 - 10d^2\mu_6 + 10d^3\mu_4 - 5d^4\mu_2 + d^5 = \tfrac{1}{2}(5\mu_8 - 5\mu_4 + 1). \tag{3.5}$$

On the other hand, ω_3 and ω_4 by (2.10) take the forms

$$\omega_3(x) = x(x^2 - e), \qquad \omega_4(x) = (x^2 - p)(x^2 - q) \quad (p < q)$$

with certain constants e, p, and q and by (1.2) satisfy

$$\int_{-1}^{1} \omega_3(x)\, xw(x)\, dx = \int_{-1}^{1} \omega_3(x)^3\, xw(x)\, dx = 0, \tag{3.6}$$

$$\int_{-1}^{1} \omega_4(x)\, w(x)\, dx = 0, \tag{3.7}$$

$$\int_{-1}^{1} \omega_4(x)\, x^2 w(x)\, dx = 0.$$

By calculation we obtain, using (3.3) and (3.4),

$$\mu_4 = e\mu_2 = \frac{e}{2}, \tag{3.8}$$

$$\mu_{10} = 3e\mu_8 - 3e^2\mu_6 + e^3\mu_4 = 3e\mu_8 + \frac{e^2}{4}(2e^2 - 9e + 3), \tag{3.9}$$

$$\mu_4 = (p+q)\,\mu_2 - pq = \frac{1}{2}(p+q) - pq, \tag{3.10}$$

$$\mu_6 = (p+q)\,\mu_4 - pq\mu_2 = \frac{1}{2}(p+q)^2 - pq(p+q) - \frac{1}{2}pq. \tag{3.11}$$

Substituting (3.10) into (3.4) and using (3.11) gives

$$2(p+q)^2 - (4pq+3)(p+q) + 4pq + 1 = 0.$$

Solving this equation with the unknown $p+q$, we obtain two solutions $p+q = 2pq + \frac{1}{2}$ and

$$p + q = 1. \tag{3.12}$$

We claim that the first solution does not satisfy (3.7). In fact, it implies $(p - \frac{1}{2})(q - \frac{1}{2}) = 0$, i.e., $p = \frac{1}{2}$ or $q = \frac{1}{2}$. But if $p = \frac{1}{2}$, say, then it follows from (3.7) and (2.13) that

$$\int_{-1}^{1} (x^2 - \tfrac{1}{2})(x^2 - 1 + q)\, w(x)\, dx = 0,$$

which, together with (3.7), yields

$$\int_{-1}^{1} (x^2 - \tfrac{1}{2})^2\, w(x)\, dx$$

$$= \tfrac{1}{2} \int_{-1}^{1} [(x^2 - \tfrac{1}{2})(x^2 - q) + (x^2 - \tfrac{1}{2})(x^2 - 1 + q)]\, w(x)\, dx = 0,$$

a contradiction.

To obtain another equation about p and q we use (1.2) and (2.10) to get

$$\int_0^1 [(x^2 - p)(x^2 - q)]^{m-1}\, w(x)\, dx = 0.$$

Substituting $x=\sqrt{y+\frac{1}{2}}$ into the above equation and using (3.12) gives

$$\int_{-1/2}^{1/2}\left(y^2+pq-\frac{1}{4}\right)^{m-1}\frac{w(\sqrt{y+\frac{1}{2}})}{\sqrt{y+\frac{1}{2}}}\,dy=0.$$

By the substitutions $y=-\sqrt{x}$ on the interval $[-\frac{1}{2}, 0]$ and $y=\sqrt{x}$ on the interval $[0, \frac{1}{2}]$, respectively, we get

$$\int_0^{1/4}\left(x+pq-\frac{1}{4}\right)^{m-1}\left[\frac{w(\sqrt{\frac{1}{2}-\sqrt{x}})}{\sqrt{x}\sqrt{\frac{1}{2}-\sqrt{x}}}+\frac{w(\sqrt{\frac{1}{2}+\sqrt{x}})}{\sqrt{x}\sqrt{\frac{1}{2}+\sqrt{x}}}\right]dx=0,$$

which holds for every $m=m_k$ by the assumptions of the theorem. Applying again Lemma 1 we get $1/4-pq=1/8$. Hence $pq=1/8$, which by (3.12) gives

$$p=\frac{2-\sqrt{2}}{4}, \qquad q=\frac{2+\sqrt{2}}{4}. \tag{3.13}$$

Then by (3.10), (3.4), (3.5), (3.8), and (3.9) we obtain $\mu_4=3/8$, $\mu_6=5/16$,

$$\mu_{10}=\frac{5}{2}\mu_8-\frac{7}{16},$$

$$\mu_{10}=\frac{9}{4}\mu_8-\frac{189}{512}.$$

The last two equations give $\mu_8=35/128$. Comparing μ_j with v_j we prove (3.1) for $j\leqslant 8$.

To prove (3.1) for all other values of j according to the assumptions of the theorem we have that for $n\geqslant 3$

$$\int_{-1}^{1} f(x)\,w(x)\,dx=\sum_{k=1}^{n} c_{0k2}f(x_k), \qquad f\in \mathbf{P}_{2n-1}, \tag{3.14}$$

$$\int_{-1}^{1} f(x)\,w(x)\,dx=\sum_{i=0}^{2}\sum_{k=1}^{n} c_{ik4}f^{(i)}(x_k), \qquad f\in \mathbf{P}_{4n-1}. \tag{3.15}$$

We claim that given $3n$ values μ_j, $j=0, 1, ..., 3n-1$, one can uniquely determine $4n$ values μ_j, $j=0, 1, ..., 4n-1$. In fact, let $\omega_n(x)=\sum_{j=0}^{n} c_jx^j$ with $c_n=1$. Substituting $f(x)=\omega_n(x)\,x^i$, $i=0, 1, ..., n-1$, into (3.14) yields

$$\sum_{j=0}^{n-1} c_j\mu_{i+j}=-\mu_{i+n}, \qquad i=0, 1, ..., n-1. \tag{3.16}$$

Since the determinant of the coefficient matrix of this system det $\{\mu_{i+j}\}_{i,j=0}^{n-1} > 0$, we can uniquely determine the solution $c_0, ..., c_{n-1}$, from which one gets its roots $x_1, ..., x_n$. Now let $A_{ik} \in \mathbf{P}_{3n-1}$ be the fundamental functions for (0, 1, 2) interpolation, i.e.,

$$A_{ik}^{(\mu)}(x_\nu) = \delta_{i\mu}\delta_{k\nu}, \qquad i, \mu = 0, 1, 2; \quad k, \nu = 1, 2, ..., n.$$

Then by (3.15)

$$c_{ik4} = \int_{-1}^{1} A_{ik}(x)\, w(x)\, x, \qquad i = 0, 1, 2; \quad k = 1, 2, ..., n, \tag{3.17}$$

which are uniquely calculated from μ_j, $j = 0, 1, ..., 3n-1$. Hence we can further calculate using (3.15), μ_j, $j = 0, 1, ..., 4n-1$. This proves our claim. According this claim using the initial 9 values $\mu_0, ..., \mu_8$, we can uniquely determine all moments $\mu_0, \mu_1, ...$ by induction, because $4n \geqslant 3(n+1)$ when $n \geqslant 3$. Since the initial 9 values of the moments and the equations (3.14)–(3.17) to determine their moments successively are the same for the weight w and the Chebyshev weight, we can obtain (3.1) and hence (1.4). ∎

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